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An Attainability Set Model for Dynamical Games

PETER E. KLOEDEN*

Mathematics Department, University of Queensland

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1. INTRODUCTION

Serious difficulties have been encountered in the literature on dynamical games, particularly on the differential level, in attempts to develop a satisfactory formulation of the game dynamics. These are associated mainly with the nature of strategies and their explicit use as input variables in the dynamics generating formula, e.g., [1, 5, 13, 18].

Control systems, which may be regarded as the dynamics of one-person games, have been successfully modelled by Roxin ([14, [15]) in terms of attainability sets as *general control systems*. In doing this he achieved considerable generality and technical simplicity by not explicitly using the control functions as input variables. Moreover, a similar attainability set formulation encompasses ordinary differential equations without uniqueness [25], contingent equations [16], and even certain stochastic control systems [6]. In this paper we will call such systems *general semi-dynamical systems* (GSDS).

Our purpose here is to use attainability sets to describe the dynamics of N-person ($N \geqslant 2$) games in such a way as to overcome, or at least circumvent, the shortcomings of currently used formulations of game dynamics. The procedure we follow inverts that usually used in the literature: To each player we assign *a priori* a set of admissible dynamical systems, all of which are general semi-dynamical systems on a complete locally compact metric state space. Strategies will appear only as indices of these admissible dynamical systems, and no assumptions will be made on the analytical nature of these strategies nor on the information patterns of the players, though it will be assumed, where applicable, that appropriate feedback mechanisms are already built into the systems.

The novelty of this construction lies in the interpretation we give to each player's admissible dynamical systems and is so chosen to avoid the subtle pitfalls introduced into the attainability concept by a conflict situation:

* Present address: Department of Mathematics, University of Melbourne.

An admissible dynamical system of a player is that corresponding to his pitting one of his admissible strategies against the totality of admissible strategies of the other $N - 1$ players.

Knowing only his own choice of strategy, such a dynamical system is a player's sharpest estimate of the future evolution of the game. If, however, he were to somehow discover the strategy (i.e., admissible dynamical system) chosen by some other player, he could considerably improve this estimate by taking the intersection of this system with his own.

In Sections 2 and 3 we state various definitions and the basic properties of general semi-dynamical systems. The difficulties encountered in formulating game dynamics are then discussed in depth in Sections 4 and 5, motivating the above construction. In Sections 6 and 7 we define the *motion-intersection* of general semi-dynamical systems in terms of their common motions (i.e., trajectories) rather than attainable points and show that it is also a general semi-dynamical system.

Four axioms are imposed on the players' sets of admissible dynamical systems in Section 8, to ensure the consistency of the above interpretation and to allow switching during the evolution of play. Then in Section 9 an example is given of a two-person differential game with no information and a consistency theorem is proved in Section 10. The advantages of this formulation of game dynamics is discussed in some detail in Section 11, with particular emphasis on its generality and conceptual simplicity.

The objectives of the players are considered in an abstract fashion in Section 12, and in Section 13 we show that our model is particularly well suited to games with qualitative objectives in that it enables each player to first analyze the game as a control rather than a conflict situation. In addition we give an example of a qualitative analog of the saddle point concept. Quantitative objectives are then discussed in Section 14 from the perspective of our model, though briefly as much of what is already known in the literature on quantitative games is valid here, except extra caution must be exercized as we do not require unique outcomes for any given strategy N-tuple.

To illustrate the potential of our model and the insight it offers into the actual conflict, we analyze a two-person quantitative game of fixed duration in the remaining sections of the paper. This game has already been considered on the contingent equation level by Krasovskii and his coworkers [10–12, 26]. Our main result is a necessary and sufficient condition for the existence of what Krasovskii calls a saddle point for this game. In terms of our model this condition is very simple to check, unlike one given by Krasovskii. Actually this game is in general nonzero sum, so we conclude with a few remarks on the applicability to it of the Nash equilibria and von Stackelberg solution concepts (e.g., [21]).

2. Notation

Let (X, d) denote a complete locally compact metric space, the points of which will represent the *states* of a given system. The independent variable $t \in R^+$ will be called *time* and point sets in X-space will be denoted by capital letters $A, B, \ldots$.

In order to avoid infinite distances between sets we will replace the given metric $d(a, b)$ by

$$\rho(a, b) = d(a, b)/(1 + d(a, b)) \tag{2.1}$$

We then define the distance between points and sets, and between sets, by

$$\rho(a, B) = \inf\{\rho(a, b);\, b \in B\}, \tag{2.2}$$

$$\delta(A, B) = \inf\{\rho(a, b);\, a \in A, b \in B\}, \tag{2.3}$$

$$\rho^*(A, B) = \sup\{\rho(a, B);\, a \in A\}, \tag{2.4}$$

$$\rho(A, B) = \max\{\rho^*(A, B), \rho^*(B, A)\}, \tag{2.5}$$

and observe that

$$\delta(A, B) \leqslant \rho^*(A, B) \leqslant \rho(A, B)$$

for any nonempty subsets A and B of X.

It is well known that (2.5) defines a pseudometric on the class of all nonempty subsets of X and a true metric, called the *Hausdorff metric*, on the class of all nonempty closed subsets of X.

3. General Semi-Dynamical Systems

A general semi dynamical system (GSDS) is given in terms of an *attainability set function* which has the following intuitive meaning: The evolution of the GSDS is determined by its initial state x_0 and initial time t_0. The set of all possible points in X that may be reached by the system at time $t_1 \geqslant t_0$ from the initial condition (x_0, t_0) is called its attainability set for (x_0, t_0, t_1) and is denoted by $F(x_0, t_0, t_1)$. The following five axioms are assumed satisfied:

Axiom 1A. $F(x_0, t_0, t_1)$ is a closed nonempty subset of X, defined for all $x_0 \in X$ and $t_1 \geqslant t_0 \geqslant 0$.

Axiom 2A (Initial Condition). $F(x_0, t_0, t_0) = \{x_0\}$ for all $x_0 \in X$ and $t_0 \geqslant 0$.

Axiom 3A (Semigroup Property).

$$F(x_0, t_0, t_2) = \bigcup \{F(x_1, t_1, t_2); x_1 \in F(x_0, t_0, t_1)\}$$
$$= F(F(x_0, t_0, t_1), t_1, t_2)$$

for all $x_0 \in X$ and $t_2 \geqslant t_1 \geqslant t_0 \geqslant 0$.

Axiom 4A. $F(x_0, t_0, t)$ is continuous in t with respect to the Hausdorff metric: given $x_0 \in X$, $t_1 \geqslant t_0 \geqslant 0$ and $\epsilon > 0$, there is a $\delta = \delta(\epsilon, x_0, t_0, t_1) > 0$ such that $| t - t_1 | < \delta$ implies that

$$\rho(F(x_0, t_0, t), F(x_0, t_0, t)) < \epsilon.$$

Axiom 5A. $F(x_0, t_0, t)$ is upper semicontinuous in (x_0, t_0, t) with respect to the Hausdorff metric: Given $x_0 \in X$, $t_1 \geqslant t_0 \geqslant 0$ and $\epsilon > 0$, there is a $\delta = \delta(\epsilon, x_0, t_0, t_1) > 0$ such that

$$\rho^*(F(x_0', t_0', t_1'), F(x_0, t_0, t_1)) < \epsilon$$

for all $x_0' \in X$ and $t_1' \geqslant t_0' \geqslant 0$ satisfying $\rho(x_0', x_0) < \delta$, $| t_0' - t_0 | < \delta$, and $| t_1' - t_1 | < \delta$.

The term *semi* in the name of these systems is a consequence of our not assuming the *backwards extendability axiom* common in much of the literature, namely, for any $x_1 \in X$ and $t_1 > t_0 \geqslant 0$, there exists an $x_0 \in X$ such that $x_1 \in F(x_0, t_0, t_1)$. Moreover, on account of the local compactness of the state space and the above axioms, it can be shown that the attainability sets are in fact compact, e.g., [7, 15].

The above attainability set formulation has been used successfully over the past twenty-five years to characterize multivalued dynamical systems of quite diverse analytical representation, such as ordinary differential equations without uniqueness [25], contingent equations [16], ordinary differential control equations [15], and certain stochastic systems [6]. Discrete-time GSDS have also been considered, e.g., [4]. (They are somewhat simpler to handle than their continuous-time counterparts and thus will be used in this paper as a source of counter examples.)

Trajectories of a GSDS can be defined in terms of its attainability set function. Actually we will reserve the name trajectory for the images of the following singlevalued functions of time which we call *motions*.

Definition 3.1. A *motion* of a GSDS is a singlevalued function $\phi: [t_0, t_1] \to X$ satisfying $\phi(t_b) \in F(\phi(t_a), t_a, t_b)$ for any $t_1 \geqslant t_b \geqslant t_a \geqslant t_0 \geqslant 0$.

Continuity of motions, not assumed in their definition, can be established from Axioms 1A–5A, as can their existence, i.e., if $x_1 \in F(x_0, t_0, t_1)$, there exists a motion ϕ: $[t_0, t_1] \to X$ with $\phi(t_0) = x_0$ and $\phi(t_1) = x_1$. It then follows from this and Axioms 1A and 3A that any motion can be prolonged for all future time $t \geqslant t_0$. We will assume that this has been done and will denote by $\Phi(x_0, t_0; F)$ the set of all motions of the GSDS F with $\phi(t_0) = x_0$. See [7]. The following result will be required several times in the sequel. It was first proved in a less general form by Barbashin and in its present form by the author [8].

THEOREM 3.1. *Let $x_n \to x_0$ in X and $t_n \to t_0$ in $[0, T]$ as $n \to \infty$ where $t_0 < T < \infty$. Furthermore, let $\{\phi_n\}$ be a sequence of motions of a* GSDS *F with $\phi_n \in \Phi(x_n, t_n; F)$ for $n = 1, 2, 3, \ldots$. Then there exists a convergent subsequence $\{\phi_{n_j}\}$ and a motion $\bar{\phi} \in \Phi(x_0, t_0; F)$ such that $\phi_{n_j}(s_{n_j}) \to \bar{\phi}(s_0)$ $j \to \infty$ for any convergent sequence $s_n \to s_0$ with $t_n \leqslant s_n \leqslant T$, $n = 1, 2, 3, \ldots$.*

We will require a topology on the space $\mathfrak{F}(X)$ of all GSDS on the state space X. For this we define the following modification of the concept of continuous convergence. It is, however, stronger because it guarantees rather than assumes in advance that the limits are unique GSDS in $\mathfrak{F}(X)$. In terms of our game model, this will ensure that the limit of a convergent net of admissible GSDS is also an admissible GSDS. We observe that the domain of definition of the GSDS in $\mathfrak{F}(X)$ is the metric space

$$Y = \bigcup \{(x_0, t_0, t_1); x_0 X, t_1 \geqslant t_0 \geqslant 0\}.$$

DEFINITION 3.2. Let $\{F_\nu\}$ be a net of GSDS in $\mathfrak{F}(X)$ and let F be a closed-set valued mapping from $Y \to X$. Then we say that F_ν converges to F, written $F_\nu \to F$, if the following four convergences

$$\rho^*(F_\nu(x_\alpha, t_\alpha, s_\alpha), F(x_0, t_0, s_0)) \to 0, \tag{3.1}$$

$$\rho(F_\nu(x_0, t_0, s_\alpha), F(x_0, t_0, s_0)) \to 0, \tag{3.2}$$

$$\rho(F_\nu(x_0, t_0, \tau_2), F(F(x_0, t_0, \tau_1), \tau_1, \tau_2)) \to 0, \tag{3.3}$$

$$\rho(F_\nu(x, t, s), F(x, t, s)) \to 0 \text{ uniformly in } (x, t, s) \in A, \tag{3.4}$$

hold simultaneously for all convergent nets $(x_\alpha, t_\alpha, s_\alpha) \to (x_0, t_0, s_0)$ in Y, all nonempty compact subsets A of Y, and all instants of time $\tau_2 \geqslant \tau_1 \geqslant t_0 \geqslant 0$.

The following result is then true [9].

THEOREM 3.2. *The convergence of definition 3.2 induces a Hausdorff topology on $\mathfrak{F}(X)$, such that $\mathfrak{F}(X)$ is a closed subspace of any larger class of mappings with this topology.*

4. Difficulties in the Formulation of Game Dynamics

A dynamical game can be defined in terms of its rules, dynamics and the objectives of its players. Quite serious difficulties, however, arise in trying to develop a satisfactory formulation for the game dynamics. The usual procedure followed in the literature (e.g., [3, 18, 23]) is to specify *a priori* classes of admissible strategies for the players and a mapping which yields the appropriate dynamics on insertion of a strategy by each of the players, e.g., in differential games this mapping in an ordinary differential equation. The difficulties encountered are most commonly associated with the analytic representation of this dynamics-generating mapping and include the existence and uniqueness of solution dynamics, the determination of maximal classes of admissible strategies for a given information pattern, and the topological properties of these classes of admissible strategies and the corresponding solution dynamics.

There have been several rather ingenious attempts on the differential level to overcome, or at least circumvent, these difficulties, and in spite of their peculiar shortcomings quite far reaching results have been obtained, e.g., [1, 5, 10]. The techniques used, however, are on the whole valid only for games with perfect information and require considerable effort to be expended on questions pertaining to the particular analytic representation used. While this may in practice be essential for the actual solution to a game, we believe it tends to obscure and detract from the fundamental purpose of a mathematical theory of dynamical games, namely to investigate the logical structure of a conflict situation.

That a differential structure is by no means essential in modeling dynamical systems has been successfully exploited in both dynamical systems and control theories. There are, however, only a few references in the literature on non-differential descriptions of game dynamics and these are either usually restricted to putsuit–evasion games or merely precede an investigation on the differential level (e.g., [3, 13, 18, 22, 23, 27]). The most promising of these seems to be the model for two-person games with perfect information advanced by Stonier [23] and Skowronski [22], in which the game dynamics are defined axiomatically in terms of a dynamical polysystem

$$F\colon \mathfrak{S}_1 \times \mathfrak{S}_2 \times X \times R^+ \times R^+ \to X,$$

where X is a complete locally compact metric state space and $\mathfrak{S}_i = \{s_i(x, t)\}$, the maximal class of admissible strategies of the ith player ($i = 1, 2$). Moreover, it is assumed for each strategy pair $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ that the mappings $F(s_1, s_2, \ldots)$, $F(s_1, \mathfrak{S}_2, \ldots)$, and $F(\mathfrak{S}_1, s_2, \ldots)$ are all general semi-dynamical systems.

This model is of considerable generality due to the diversity of systems describable as general semi-dynamical systems and, heuristically, subsumes differential treatments of game dynamics, in particular the contingent equation work of the Krasovskii group (e.g., [10–12, 26]). The question of existence is avoided here by assumption and the possible nonuniqueness of outcomes for a strategy pair (s_1, s_2) is handled in a most natural way. There are, however, severe restrictions due to the explicit use of strategies as input variables in the mapping F. These are mainly the need to topologize the classes $\mathfrak{S}_i$ $(i = 1, 2)$ of admissible strategies and to impose some kind of continuity condition on F with respect to these strategies. An example of this is Stonier's use of continuous nonempty compact setvalued mappings as admissible strategies and his assumption that F is upper semicontinuous in them.

Control systems may be regarded as the dynamics of one-person games. When Roxin [14] developed his general (semi) dynamical systems formulation of them, he achieved considerable generality and technical simplicity by not explicitly using the control functions as input variables. Moreover, both Petrosyan [13] and Varaiya [27] have in effect modeled the dynamics of pursuit–evasion games with two such uncoupled control systems and achieved much the same benefits. We believe they are also to be had for dynamical games in general provided the strategies are assumed already built into the game dynamics and are used only to index the corresponding dynamics.

5. Attainability Sets in Dynamical Games

Subtle nuances are introduced into the attainability concept on passing from control systems to dynamical games with more than one player, due to the conflicting aims of the players or deficiencies in communication between them.

In a control system there exists by definition for any attainable point $x_1 \in F(x_0, t_0, t_1)$ an admissible control which the controller can implement to ensure that x_1 is reached at time t_1. In a dynamical game, however, no one player generally has complete control over the game dynamics. In particular, in a two-person game to fix ideas, if Player 1 selects an admissible strategy $s_1 \in \mathfrak{S}_1$, then, in the absence of knowledge of the other player's choice of strategy, he can predict the future evolution of play starting at (x_0, t_0) only to within the attainability set $F(s_1, \mathfrak{S}_2, x_0, t_0, t)$ in the terminology of the Stonier–Skowronski model. Which point or points are actually reached in this set depends very much on the other player's choice of strategy. Roxin [20] has quite lucidly illustrated the pitfalls that may arise in assuming any more than this.

Nevertheless we can use this apparent restriction in a most natural way to axiomatically construct a model of game dynamics without having to explicitly use the strategies as input variables. To do this we assign to each player, i ($i = 1, 2$), a set $\mathfrak{D}_i$ of *admissible general semi-dynamical systems* in $\mathfrak{F}(X)$. We interpret each dynamic system in $\mathfrak{D}_i$ as the dynamics resulting from his "playing" one of his admissible strategies against the totality of admissible strategies of the other player. The set $\mathfrak{S}_i$ of indices of the GSDS in $\mathfrak{D}_i$ will be regarded as the set of admissible strategies of the ith player ($i = 1, 2$). Thus, for example, a GSDS F_{s_1} in $\mathfrak{D}_i$ corresponds to the GSDS $F(s_1, \mathfrak{S}_2, \ldots)$ in the Stonier–Skowronski model.

An interesting aspect of this formulation is that, in games in which the players have qualitative objectives, it enables each player to first analyze the game as a control problem. For this he searches his set $\mathfrak{D}_i$ for an admissible GSDS which fulfils his desired goal; if successful he can then play the game without having to take into account the possible behavior of the other player.

To proceed further with the construction of our game model we need somehow to be able to reclaim from the respective choices $F_{s_1} \in \mathfrak{D}_1$ and $F_{s_2} \in \mathfrak{D}_2$ of admissible GSDS the dynamics corresponding to the strategy pair (s_1, s_2). In the context of the Stonier–Skowronski model this means the GSDS $F(s_1, s_2, \ldots)$. For this we need to define an intersection of the two GSDS F_{s_1} and F_{s_2} in such a way as to yield a third GSDS $F_{s_1 s_2}$ representing the dynamics corresponding to the player's using strategies s_1 and s_2, respectively.

6. The Intersection of GSDS

Superficially the most obvious way of defining the intersection F_{12} of two GSDS F_1 and F_2 on the same state space X is as the intersection of their attainability sets,

$$F_{12}(x_0, t_0, t) = F_1(x_0, t_0, t) \cap F_2(x_0, t_0, t),$$

which we assume nonempty for all $x_0 \in X$ and $t \geqslant t_0$.

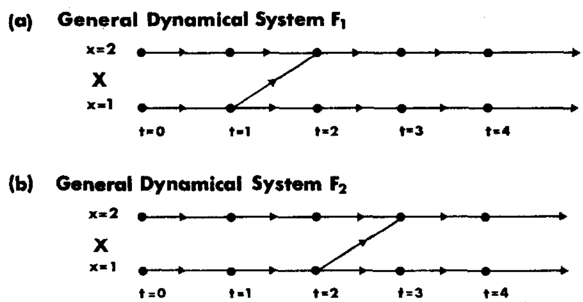

Figure 1

This is, however, quite unsatisfactory because $F_{12}(x_0, t_0, t)$ thus defined need not satisfy the semigroup property (Axiom 3A) of a GSDS as we show in the following example.

EXAMPLE 6.1. Consider the two discrete-time GSDS F_1 and F_2 on the discrete state space $X = \{1, 2\}$ which are defined graphically in Figs. 1(a) and 1(b), respectively. Then with F_{12} defined as above we have

$$F_{12}(1, 2, j) = \{1\} \qquad \text{for} \quad j = 2, 3, 4, \ldots,$$

and

$$\begin{aligned} F_{12}(1, 0, j) &= \{1\} \qquad \text{for} \quad j = 0, 1 \text{ and } 2, \\ &= X \qquad \text{for} \quad j = 3, 4, 5, \ldots, \end{aligned}$$

which means that

$$\begin{aligned} F_{12}(F_{12}(1, 0, 2), 2, j) &\equiv F_{12}(1, 2, j) \\ &\subsetneqq F_{12}(1, 0, j) \qquad \text{for} \quad j = 3, 4, 5, \ldots . \end{aligned}$$

This situation is intuitively quite absurd if we are to interpret F_{12} as a dynamical system because it implies that this system has "attainable" points which cannot be reached by a trajectory of the system. We can avoid it if instead we define the intersection of GSDS in terms of the motions (trajectories) that they have in common rather than attainable points. To distinguish this from the above intersection of attainability sets we call it their *motion-intersection*.

DEFINITION 6.1. Let F_1 and F_2 be two GSDS on a state space X such that for each $x_0 \in X$ and $t_0 \geqslant 0$

$$\bigcap_{i=1}^{2} \Phi(x_0, t_0) \neq \varnothing. \tag{6.1}$$

Then their *motion-intersection* F_{12} is defined for each $x_0 \in X$ and $t_1 \geqslant t_0 \geqslant 0$ as

$$F_{12}(x_0, t_0, t_1) = \bigcup \left\{ x_1 \in X; \exists \phi \in \bigcap_{i=1}^{2} \Phi(x_0, t_0; F_i) \text{ with } \phi(t_1) = x_1 \right\}.$$

In order to show that this motion-intersection, where defined, is also a GSDS, we will require the following lemma, which is an immediate consequence of Theorem 3.1.

LEMMA 6.1. *Let F_1 and F_2 be two GSDS as in Definition 6.1 and let $x_n \to x_0$ in X and $t_n \to t_0$ in $[0, T]$ as $n \to \infty$ where $t_0 < T < \infty$. Furthermore, let $\{\phi_n\}$ be a sequence of motions with $\phi_n \in \bigcap_{i=1}^2 \Phi(x_n, t_n; F_i)$ for $n = 1, 2, 3, \ldots$. Then there exists a convergent subsequence $\{\phi_{n_j}\}$ and a motion $\bar{\phi} \in \bigcap_{i=1}^2 \Phi(x_0, t_0; F_i)$ such that $\phi_{n_j}(s_{n_j}) \to \bar{\phi}(s_0)$ as $j \to \infty$ for any convergent sequence $s_n \to s_0$ with $t_n \leqslant s_n \leqslant T$, $n = 1, 2, 3, \ldots$.*

THEOREM 6.1. *The motion-intersection of two GSDS, where defined, is also a GSDS.*

Proof. Let F_1 and F_2 be two GSDS as in Definition 6.1. We will show with the following five propositions that their motion-intersection F_{12} is also a GSDS.

PROPOSITION 6.1. *$F_{12}(x_0, t_0, t_1)$ is a nonempty compact subset of X for each $x_0 \in X$ and $t_1 \geqslant t_0 \geqslant 0$.*

Proof. Nonemptiness follows from (6.1) and the prolongability for all future time of the motions of a GSDS. Now let $x_n \in F_{12}(x_0, t_0, t_1)$ for $n = 1, 2, 3, \ldots$. Then there exist motions $\phi_n \in \bigcap_{i=1}^2 \Phi(x_0, t_0; F_i)$ with $\phi_n(t_1) = x_n$ for $n = 1, 2, 3, \ldots$ and by Lemma 6.1 there is a convergent subsequence $\phi_{n_j} \to \bar{\phi} \in \bigcap_{i=1}^2 \Phi(x_0, t_0; F_i)$ as $j \to \infty$. Hence, there is a convergent subsequence $x_{n_j} = \phi_{n_j}(t_1) \to x = \bar{\phi}(t_1) \in F_{12}(x_0, t_0, t_1)$, from which we have the compactness of $F_{12}(x_0, t_0, t_1)$. ∎

PROPOSITION 6.2. *$F(x_0, t_0, t_0) = \{x_0\}$ for each $x_0 \in X$ and $t_0 \geqslant 0$.*

Proof. By definition $\phi(t_0) = x_0$ for each $\phi \in \bigcap_{i=1}^2 \Phi(x_0, t_0; F_i)$. ∎

PROPOSITION 6.3. *For any $x_0 \in X$ and $t_2 \geqslant t_1 \geqslant t_0 \geqslant 0$,*

$$F_{12}(x_0, t_0, t_2) = F_{12}(F_{12}(x_0, t_0, t_1), t_1, t_2).$$

Proof. We prove this in two parts, which can be combined to give the desired result.

(A) Let $x_2 \in F_{12}(F_{12}(x_0, t_0, t_1), t_1, t_2)$. Then there exists an $x_1^* \in F_{12}(x_0, t_0, t_1)$ such that $x_2 \in F_{12}(x_1^*, t_1, t_2)$, and consequently, there exist motions

$$\phi_1 \in \bigcap_{i=1}^{2} \Phi(x_0, t_0; F_i) \qquad \text{and} \qquad \phi_2 \in \bigcap_{i=1}^{2} \Phi(x_1^*, t_1; F_i)$$

such that $x_1^* = \phi_1(t_1)$ and $x_2 = \phi_2(t_2)$.

Now define

$$\bar{\phi}(t) = \phi_1(t) \quad \text{for} \quad t_0 \leqslant t \leqslant t_1,$$
$$= \phi_2(t) \quad \text{for} \quad t_1 \leqslant t.$$

Then by Axiom 3A, $\bar{\phi} \in \Phi(x_0, t_0; F_i)$ for $i = 1$ and 2, and is thus contained in the intersection of these two sets. By Definition 6.1, we thus have $x_2 = \phi_2(t_2) \in F_{12}(x_0, t_0, t_2)$, and as it was otherwise arbitrarily chosen

$$F_{12}(F_{12}(x_0, t_0, t_1), t_1, t_2) \subset F_{12}(x_0, t_0, t_2).$$

(B) For any $x_2 \in F_{12}(x_0, t_0, t_2)$, there exists a motion $\phi \in \bigcap_{i=1}^{2} \Phi(x_0, t_0; F_i)$ with $x_2 = \phi(t_2)$. By Axiom 3A $\phi \in \Phi(\phi(t), t; F_i)$ for each $t \geqslant t_0$ and $i = 1$ and 2. Thus, $\phi \in \bigcap_{i=1}^{2} \Phi(x_1, t_1; F_i)$ where $x_1 = \phi(t_1) \in F_{12}(x_0, t_0, t_1)$, and so $x_2 = \phi(t_2) \in F_{12}(x_1, t_1, t_2) \subset F_{12}(F_{12}(x_0, t_2, t_1), t_1, t_2)$. As x_2 was otherwise arbitrary, we have

$$F_{12}(x_0, t_0, t_2) \subset F_{12}(F_{12}(x_0, t_0, t_1), t_1, t_2). \quad \blacksquare$$

PROPOSITION 6.4. *$F_{12}(x_0, t_0, s_0)$ is upper semicontinuous in (x_0, t_0, s_0) with respect to the Hausdorff metric.*

Proof. Let $(x_n, t_n, s_n) \to (x_0, t_0, s_0)$ in Y as $n \to \infty$ and suppose that

$$\lim_{n\to\infty} \rho^*(F_{12}(x_n, t_n, s_n), F_{12}(x_0, t_0, s_0)) \neq 0.$$

Then there can be found a subsequence $\{x_{n_j}, t_{n_j}, s_{n_j}\}$ and a real number $\alpha > 0$ such that for $j = 1, 2, 3,...,$

$$\rho^*(F_{12}(x_{n_j}, t_{n_j}, s_{n_j}), F_{12}(x_0, t_0, s_0)) \geqslant \alpha > 0 \tag{6.2}$$

Now by compactness there exist $y_{n_j} \in F_{12}(x_{n_j}, t_{n_j}, s_{n_j})$ for $j = 1, 2, 3,...$ such that

$$\begin{aligned} &\rho^*(F_{12}(x_{n_j}, t_{n_j}, s_{n_j}), F_{12}(x_0, t_0, s_0)) \\ &\quad = \rho(y_{n_j}, F_{12}(x_0, t_0, s_0)) \leqslant \rho(y_{n_j}, y) \end{aligned} \tag{6.3}$$

for any $y \in F_{12}(x_0, t_0, s_0)$. Then by Definition 6.1 there exist motions $\phi_{n_j} \in \bigcap_{i=1}^{2} \Phi(x_0, t_0; F_i)$ with $\phi_{n_j}(s_{n_j}) = y_{n_j}$ for $j = 1, 2, 3,...$, and hence, by Lemma 6.1 there is a convergent subsequence $\phi_{n_{j_k}} \to \bar{\phi} \in \bigcap_{i=1}^{2} \Phi(x_0, t_0; F_i)$ in such a way that $\phi_{n_{j_k}}(s_{n_{j_k}}) \to \bar{\phi}(s_0)$ as $k \to \infty$.

Consequently by (6.2) and (6.3) we have for k sufficiently large

$$\begin{aligned} \rho(\phi_{n_{j_k}}(s_{n_{j_k}}), \bar{\phi}(s_0)) &\geqslant \rho^*(F_{12}(x_{n_{j_k}}, t_{n_{j_k}}, s_{n_{j_k}}), F_{12}(x_0, t_0, s_0)) \\ &\geqslant \alpha > 0, \end{aligned}$$

which is absurd. From this contradiction we have the validity of the proposition. ▮

PROPOSITION 6.5. *$F_{12}(x_0, t_0, t)$ is continuous in t with respect to the Hausdorff metric.*

Proof. In view of Proposition 6.4 it remains only to prove for any $t_0 \leqslant s_n \to s_0$ that

$$\lim_{n\to\infty} \rho^*(F_{12}(x_0, t_0, s_0), F_{12}(x_0, t_0, s_n)) = 0.$$

If this were not so there would exist a subsequence $\{s_{n_j}\}$ and a real number $\alpha > 0$ such that for $j = 1, 2, 3,...$

$$\rho^*(F_{12}(x_0, t_0, s_0), F_{12}(x_0, t_0, s_{n_j})) \geqslant \alpha > 0. \tag{6.4}$$

By compactness there exist $z_{n_j} \in F_{12}(x_0, t_0, s_0)$ such that

$$\rho^*(F_{12}(x_0, t_0, s_0), F_{12}(x_0, t_0, s_{n_j})) = \rho(z_{n_j}, F_{12}(x_0, t_0, s_{n_j})) \text{ for } j = 1, 2, 3,...$$

and then by Definition 6.1, there exist motions $\phi_{n_j} \in \bigcap_{i=1}^{2} \Phi(x_0, t_0; F_i)$ with $\phi_{n_j}(s_0) = z_{n_j}$ for $j = 1, 2, 3,....$ Thus,

$$\begin{aligned}\rho^*(F_{12}(x_0, t_0, s_0), F_{12}(x_0, t_0, s_{n_j}))\\ = \rho(\phi_{n_j}(s_0), F_{12}(x_0, t_0, s_{n_j})) \leqslant \rho(\phi_{n_j}(s_0), \phi(s_{n_j})).\end{aligned} \tag{6.5}$$

From Lemma 6.1 there is a convergent subsequence

$$\phi_{n_{j_k}} \to \bar{\phi} \in \bigcap_{i=1}^{2} \Phi(x_0, t_0; F_i)$$

such that $\phi_{n_{j_k}}(s_{n_{j_k}}) \to \bar{\phi}(s_0)$ as $k \to \infty$. Consequently

$$\rho(\phi_{n_{j_k}}(s_0), \phi_{n_{j_k}}(s_{n_{j_k}})) \leqslant \rho(\phi_{n_{j_k}}(s_0), \bar{\phi}(s_0)) + \rho(\bar{\phi}(s_0), \phi_{n_{j_k}}(s_{n_{j_k}})) \to 0 \quad \text{as} \quad k \to \infty,$$

which contradicts (6.4) and (6.5). This proves the proposition. ▮

We have thus completed the proof to Theorem 6.1 and have as a corollary:

COROLLARY 6.1. *For any $x_0 \in X$ and $t_1 \geqslant t_0 \geqslant 0$*

(i) $\Phi(x_0, t_0; F_{12}) = \bigcap_{i=1}^{2} \Phi(x_0, t_0; F_i)$

(ii) $F_{12}(x_0, t_0, t_1) \subset \bigcap_{i=1}^{2} F_{12}(x_0, t_0, t_1)$ *where this inclusion may be strict.*

Our reason for not assuming the backwards extendability axiom in this paper is that it need not be preserved in the motion-intersection, even though the original two systems satisfy it.

EXAMPLE 6.2. Consider the two discrete-time GSDS F_1 and F_2 on the discrete state space $X = \{1, 2, 3\}$, which are defined graphically in Figs. 2(a) and 2(b) respectively. They both satisfy the backwards extendability axiom. Their motion-intersection F_{12} is shown graphically in Fig. 2(c). It satisfies Axioms 1A–5A of a GSDS, but the backwards extendability axiom is violated at $(x_1, t_1) = (2, 2)$.

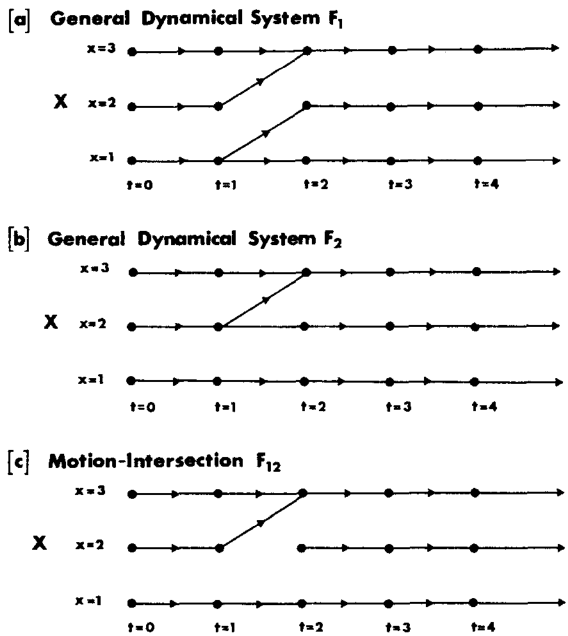

FIGURE 2

7. A CONVENIENT NOTATION

We have so far considered the motion-intersection of only two GSDS. In order to extend it to more than two, we introduce a convenient notation describing it as a binary operation on $\mathfrak{F}(X)$.

DEFINITION 7.1. Let F_1 and F_2 be two GSDS as in Definition 6.1, and let F_{12} be their motion-intersection.

Then the binary operation $\wedge$ of motion-intersection on $\mathfrak{F}(X)$ is defined for F_1 and F_2 as

$$F_{12} = F_1 \wedge F_2 = F_2 \wedge F_1 ,$$

i.e., where defined $\wedge$ is commutative.

PROPOSITION 7.1. *Let F_1, F_2 and F_3 be three GSDS in $\mathfrak{F}(X)$. Then a necessary and sufficient condition for the existence of $(F_1 \wedge F_2) \wedge F_3$ and $F_1 \wedge (F_2 \wedge F_3)$ is that $\bigcap_{i=1}^{3} \Phi(x_0, t_0; F_i) \neq \varnothing$. Furthermore, we then have $(F_1 \wedge F_2) \wedge F_3 \equiv F_1 \wedge (F_2 \wedge F_3)$, and thus write it as $F_1 \wedge F_2 \wedge F_3$ or $\bigwedge_{i=1}^{3} F_i$, i.e., $\wedge$ is associative where defined.*

Proof. This follows immediately from the various definitions involved. ∎

Thus, for $N \geqslant 2$, GSDS $F_1, F_2, \ldots, F_N$ in $\mathfrak{F}(X)$ with $\bigcap_{i=1}^{N} \Phi(x_0, t_0; F_i) \neq 0$ for all $x_0 \in X$ and $t_0 \geqslant 0$, we can define their motion-intersection as

$$F_{12\cdots N} = F_1 \wedge F_2 \wedge \cdots \wedge F_N = \bigwedge_{i=1}^{N} F_i .$$

Clearly it is also a GSDS in $\mathfrak{F}(X)$, and for all $x_0 \in X$ and $t_0 \geqslant 0$

$$\Phi(x_0, t_0; F_{12\cdots N}) = \bigcap_{i=1}^{N} \Phi(x_0, t_0; F_i).$$

8. AN AXIOMATIC MODEL OF GAME DYNAMICS

We will now axiomatically develop an attainability set model of game dynamics as outlined in Section 5, though here we will consider $N \geqslant 2$ players. In what follows we invert the standard practice of specifying classes of admissible strategies and a dynamics-generating mapping, by considering the admissible dynamics themselves to be the fundamental entities of a dynamical game. We will introduce strategies only as indices of these admissible dynamics, which will be described entirely by means of general semi-dynamical systems. It should be recalled that $\mathfrak{F}(X)$ denotes the Hausdorff topological space of all general semi-dynamical systems on a complete locally compact metric state space X.

Each player i $(i = 1, 2, \ldots, N)$ is assigned *a priori* a set $\mathfrak{D}_i \subset \mathfrak{F}(X)$ of *admissible dynamical systems* satisfying

AXIOM I. *The set $\mathfrak{D}_i$ of admissible dynamical systems of the ith player $(i = 1, 2, \ldots, N)$ is a nonempty compact subset of $\mathfrak{F}(X)$.*

These sets $\mathfrak{D}_i$ will each be indexed by a set $\mathfrak{S}_i$ of what we call *admissible strategies* of the *i*th player and for any $s_i \in \mathfrak{S}_i$, we denote the corresponding

GSDS by F_{s_i} $(i = 1, 2, \ldots, N)$. We assume as part of the rules of the game that each player knows all of the sets $\mathfrak{D}_i$ $(i = 1, 2, \ldots, N)$ of admissible dynamical systems, that is, not only his own but also those of the other $N - 1$ players. However, we make no explicit assumptions whatsoever about the nature of the information of state available to the players, although we assume wherever applicable that the appropriate feedback mechanism is already built into the admissible dynamical systems.

The crux of the model lies in the interpretation that we give to the admissible dynamical systems of the players:

An admissible dynamical system $F_{s_i} \in \mathfrak{D}_i$ corresponds to the dynamics resulting when the ith *player uses his admissible strategy $s_i \in \mathfrak{S}_i$ against the totality of all possible admissible strategies of the other $N - 1$ players $(i = 1, 2, \ldots, N)$.*

For example, in the context of the Stonier–Skowronski model where $N = 2$, for some admissible strategy $s_1 \in \mathfrak{S}_1$, the admissible dynamical system $F_{s_i} \in \mathfrak{D}_1$ corresponds to the GSDS $F(s_1, \mathfrak{S}_2, \ldots)$. To ensure the consistency of this interpretation we must include the following axiom in our model:

Axiom II. *For all $x_0 \in X$, $t_0 \geqslant 0$ and $i, j = 1, 2, \ldots, N$ with $i \neq j$,*

$$\bigcup \{\Phi(x_0, t_0; F_{s_i}); s_i \in \mathfrak{S}_i\} = \bigcup \{\Phi(x_0, t_0; F_{s_j}); s_j \in \mathfrak{S}_j\}.$$

We denote the union of motions in Axiom II by $\Phi(x_0, t_0)$ and observe that it corresponds to the totality of all motions of the game emanating from (x_0, t_0) for all possible strategy N-tuples $(s_1, s_2, \ldots, s_N) \in \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_N$ of the N players.

We justify our assigning a player's admissible dynamics in terms of the sets $\mathfrak{D}_i \subset \mathfrak{F}(X)$ as interpreted above by the fact that if he does not know which strategies the other $N - 1$ players have chosen, then Player i can predict the future course of play starting at (x_0, t_0) only to within the set $F_{s_i}(x_0, t_0, t)$ for each future instant of time $t \geqslant t_0$ and his own choice of strategy $s_i \in \mathfrak{S}_i$. If, however, he discovers by some means (e.g., spying on antagonists or communicating with coalition partners) the strategy choice of some other player, he should be able to considerably sharpen his estimate of the future evolution of the *partie*. This can be done quite simply and naturally by taking the motion-intersection of his chosen admissible GSDS F_{s_i} with the admissible GSDS F_{s_j} that he has discovered player j $(\neq i)$ to have chosen. To allow for such motion-intersections we need the following axiom in our model:

Axiom III. *For any admissible GSDS $F_{s_i} \in \mathfrak{D}_i$ $(i = 1, 2, \ldots, N)$ and $x_0 \in X$, $t_0 \geqslant 0$,*

$$\bigcap_{i=1}^{N} \Phi(x_0, t_0; F_{s_i}) \neq \varnothing.$$

Now let $(i_1, i_2, \dots, i_N)$ be any permutation of $(1, 2, \dots, N)$ and let $j = 2, 3, \dots, N$. Then by Axiom III for any $F_{s_{i_k}} \in \mathfrak{D}_{i_k}$ $(k = 1, 2, \dots, j)$, the motion-intersection

$$F_{s_{i_1} s_{i_2} \cdots s_{i_k}} = \bigwedge_{k=1}^{j} F_{s_{i_k}}$$

is defined and is a GSDS on X. For $j = 1, 2, \dots, N-1$ it corresponds to the dynamics resulting when the j players $(i_1, i_2, \dots, i_j)$ play the strategy j-tuple $(s_{i_1}, s_{i_2}, \dots, s_{i_j}) \in \mathfrak{S}_{i_1} \times \mathfrak{S}_{i_2} \times \cdots \times \mathfrak{S}_{i_j}$, not necessarily cooperatively, against the totality of all possible admissible strategies of the remaining $N - j$ players $(i_{j+1}, i_{j+2}, \dots, i_N)$. When $j = N$ such an $F_{s_{i_1} s_{i_2} \cdots s_{i_N}}$ corresponds to the dynamics resulting from the i_kth player using strategy $s_{i_k} \in \mathfrak{S}_{i_k}$ for $k = 1, 2, \dots, N$.

Since the operation $\wedge$ of motion-intersection is both associative and commutative, where defined, these $F_{s_{i_1} s_{i_2} \cdots s_{i_j}}$ are independent of the particular ordering $(i_1, i_2, \dots, i_j)$. This suggests we could perhaps use a canonical ordering such as $i_1 < i_2 < \cdots < i_j$. Then for each $j = 2, 3, \dots, N$ and such an ordering, by taking all of the appropriate motion-intersections, we can construct the sets

$$\mathfrak{D}_{i_1 i_2 \cdots i_j} = \{F_{s_{i_1} s_{i_2} \cdots s_{i_j}};\ s_{i_k} \in \mathfrak{S}_{i_k} \text{ for } k = 1, 2, \dots, j\}$$

of admissible dynamical systems of the *syndicate* (not necessarily cooperative) or *coalition* (if cooperative) of players $(i_1, i_2, \dots, i_j)$. (When $j \neq N$ here we make no assumptions as to whether or not the remaining $N - j$ players form a counter-coalition.)

The very name ***admissible dynamical systems of the*** i**th** ***player*** for the set $\mathfrak{D}_i \subset \mathfrak{F}(X)$ suggests that if he so desires, Player i can implement any GSDS in $\mathfrak{D}_i$ in any *partie* $(i = 1, 2, \dots, N)$. To enable him to switch to a new admissible dynamical system during its progress and in such a way that the composite dynamical system is also admissible, we include the following *switching axiom* in our model:

Axiom IV. *For any* $F_{s_i'}, F_{s_i''} \in \mathfrak{D}_i$ $(i = 1, 2, \dots, N)$ *and* $x_0 \in X$, $t_1 > t_0 \geqslant 0$, *there exists an* $F_{\bar{s}_i} \in \mathfrak{D}_i$ *such that*

$$\begin{aligned} F_{\bar{s}_i}(x_0, t_0, t) &= F_{s_i'}(x_0, t_0, t) && \text{for} \quad t_0 \leqslant t \leqslant t_1, \\ &= F_{s_i''}(F_{s_i'}(x_0, t_0, t_1), t_1, t) && \text{for} \quad t_1 \leqslant t. \end{aligned}$$

In terms of admissible strategies Axiom IV means that the piecewise composition of admissible strategies is also an admissible strategy.

9. An Example

We now give a simple example to illustrate our attainability set model of game dynamics. It is based on a two-person differential game with no information (e.g., see [28]), for which we will explicitly derive the sets of admissible general semi-dynamical systems for each player. It can be easily seen by inspection that they satisfy the four axioms of the model. As for the time being we are only interested in the game dynamics, we will not specify objectives for the players here. We will, however, do that later on in the paper.

Example 9.1. We consider on the state space $X = R$ a two-person differential game with no information (other than initial conditions) governed by the ordinary differential equation

$$\dot{x} = (-u + v)x \tag{9.1}$$

where $0 \leqslant u \leqslant 1$ and $0 \leqslant v \leqslant 1$.

As the players have no information of state other than the initial condition, their admissible strategies are respectively the Lebesgue-measurable open-loop control functions $u\colon R^+ \to [0, 1]$ a.e. and $v\colon R^+ \to [0, 1]$ a.e. We will denote these sets of admissible strategies by $\mathfrak{U}$ and $\mathfrak{V}$, respectively. Then the sets $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of admissible general semi-dynamical systems are constructed as follows:

For any $u \in \mathfrak{U}$, $F_u \in \mathfrak{D}_1$ is the GSDS satisfying the contingent equation

$$\dot{x}(t) \in [-u(t), 1 - u(t)]\, x(t)$$

and can be written for any $x_0 \geqslant 0$ and $t \geqslant t_0 \geqslant 0$ as

$$F_U(x_0, t_0, t) = \left[x_0 \exp\left(-\int_{t_0}^{t} u(s)\, ds\right), x_0 \exp\left(t - t_0 - \int_{t_0}^{t} u(s)\, ds\right)\right].$$

Similarly, for any $v \in \mathfrak{V}$, $F_v \in \mathfrak{D}_2$ is the GSDS satisfying the contingent equation

$$\dot{x}(t) \in [v(t) - 1, v(t)]\, x(t),$$

and for any $x_0 \geqslant 0$ and $t \geqslant t_0 \geqslant 0$ can be written as

$$F_v(x_0, t_0, t) = \left[x_0 \exp\left(-t + t_0 + \int_{t_0}^{t} v(s)\, ds\right), x_0 \exp\left(\int_{t_0}^{t} v(s)\, ds\right)\right].$$

In both cases when $x_0 < 0$ the attainability sets are the same as above, but with the endpoints interchanged. Also it is easy to see that the two sets $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of admissible general semi-dynamical systems satisfy Axioms I–IV of the game model. Moreover, we observe that the motion-intersections $F_{uv} \in \mathfrak{D}_{12}$ for any $u \in \mathfrak{U}$ and $v \in \mathfrak{V}$ each have a unique motion for any initial condition (x_0, t_0). In fact,

$$F_{uv}(x_0, t_0, t) = \left\{ x_0 \exp \left(\int_{t_0}^{t} (-u(s) + v(s))\, ds \right) \right\},$$

which is of course the unique solution through (x_0, t_0) of (9.1). This uniqueness is, however, peculiar to this particular example and is not demanded by Axioms I–IV.

10. A Consistency Theorem

Axioms II and III were included in our model for the consistency of our interpretation of the admissible general semi-dynamical systems. The former could have been expressed, possibly more naturally, in terms of attainability sets:

Axiom II*. *For all* $x_0 \in X$, $t_1 \geqslant t_0 \geqslant 0$ *and* $i, j = 1, 2, \ldots, N$ *with* $i \neq j$

$$\bigcup \{F_{s_i}(x_0, t_0, t_1);\, s_i \in \mathfrak{S}_i\} \equiv \bigcup \{F_{s_j}(x_0, t_0, t_1);\, s_j \in \mathfrak{S}_j\}.$$

In view of our definition of motion-intersection, Axiom III seems best left in its present form. From these two axioms we now show the intuitive consistency of the way in which we reclaim the admissible dynamical systems for syndicates of players.

Theorem 10.1. *Let* $k = 2, 3, \ldots, N$. *Then for any* $x_0 \in X$, $t_0 \geqslant 0$ *and* $F_{s_{i_j}} \in \mathfrak{D}_{i_j}$ $(j = 1, 2, \ldots, k-1)$,

$$\Phi(x_0, t_0; F_{s_{i_1} s_{i_2} \cdots s_{i_{k-1}}}) \equiv \bigcup \{\Phi(x_0, t_0; F_{s_{i_1} s_{i_2} \cdots s_{i_{k-1}} s_{i_k}});\, s_{i_k} \in \mathfrak{S}_{i_k}\}.$$

Proof. All of these motion-intersections are defined by Axiom III and our remarks in Sections 6 and 7. Then for any $s_{i_k} \in \mathfrak{S}_{i_k}$

$$\begin{aligned}
\Phi(x_0, t_0; F_{s_{i_1} s_{i_2} \cdots s_{i_k}}) &= \bigcap_{j=1}^{k} \Phi(x_0, t_0; F_{s_{i_j}}) \\
&= \Phi(x_0, t_0; F_{s_{i_k}}) \cap \left(\bigcap_{j=1}^{k-1} Q(x_0, t_0; F_{s_{i_j}}) \right) \\
&\subset \bigcap_{j=1}^{k-1} \Phi(x_0, t_0; F_{s_{i_j}}) \\
&= \Phi(x_0, t_0; F_{s_{i_1} s_{i_2} \cdots s_{i_{k-1}}}),
\end{aligned}$$

and thus,

$$\bigcup \{\Phi(x_0, t_0; F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}s_{i_k}}); s_{i_k} \in \mathfrak{S}_{i_k}\} \subset \Phi(x_0, t_0; F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}}).$$

Now take any $\phi \in \Phi(x_0, t_0; F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}})$. Then by Axiom II,

$$\begin{aligned}\Phi(x_0, t_0; F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}}) &= \bigcap_{j=1}^{k-1} \Phi(x_0, t_0; F_{s_{i_k}}) \\ &\subset \Phi(x_0, t_0) \\ &= \bigcup \{\Phi(x_0, t_0; F_{s_{i_k}}); s_{i_k} \in \mathfrak{S}_k\},\end{aligned}$$

so there exists an $s_{i_k}^* \in \mathfrak{S}_{i_k}$ such that $\phi \in \Phi(x_0, t_0; F_{s_{i_k}^*})$. Hence,

$$\begin{aligned}\phi \in \Phi(x_0, t_0; F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}}) &\cap \Phi(x_0, t_0; F_{s_{i_k}^*}) \\ &= \Phi(x_0, t_0; F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}s_{i_k}^*}) \\ &\subset \bigcup \{\Phi(x_0, t_0; F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}s_{i_k}}); s_{i_k} \in \mathfrak{S}_{i_k}\},\end{aligned}$$

and as ϕ was otherwise arbitrary we have

$$\Phi(x_0, t_0; F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}}) \subset \bigcup \{\Phi(x_0, t_0; F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}s_{i_k}}); s_{i_k} \in \mathfrak{S}_{i_k}\}.$$

This completes the proof of the theorem. ▮

From this result and Definition 6.1 of motion-intersection, we thus have

COROLLARY 10.1 *Let $k = 2, 3,\ldots, N$. Then for any $x_0 \in X$, $t_1 \geqslant t_0 \geqslant 0$ and $F_{s_{i_j}} \in \mathfrak{D}_{i_j}$ $(j = 1, 2,\ldots, k-1)$,*

$$F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}}(x_0, t_0, t_1) \equiv \bigcup \{F_{s_{i_1}s_{i_2}\cdots s_{i_{k-1}}s_{i_k}}(x_0, t_0, t_1); s_{i_k} \in \mathfrak{S}_{i_k}\}.$$

11. Discussion of the Model

The axiomatic model of game dynamics presented above fully exploits the benefits of an abstract theory of dynamical systems and overcomes, or at least circumvents, many of the obstacles and shortcomings associated with contemporary formulations of game dynamics. (The reader is also referred to the appendix at the end of the paper.) Since the conceptual simplicity and generality of general semi-dynamical systems and, in particular, the unifying approach they offer for a diversity of systems and state spaces are well known, we will restrict our comments here to the game-theoretic aspects of our model.

Our attainability set formulation of game dynamics has utilized in a most natural and unconfusing way the concept of attainability in conflict situations. In addition it enables players with qualitative objectives to first investigate the game as a control problem. The assumptions made are of considerable generality and do not restrict the model to any particular number of players or class of dynamical games, as do for instance the pursuit-evasion models of Petrosyan [13] and Varaiya [27]. (These can be considered in the context of our model by taking as X the cartesian product of the pursuer and evader state spaces.) Moreover, the model can be easily modified by use of Roxin's *local general semi-dynamical systems* [17] to include dynamical games on a restricted state space Z, where Z is a closed subspace of X.

The most significant characteristic of our model is the total absence of *a priori* assumptions on the nature of the information patterns, with respect to the state-space X, of the players. This is illustrated in its subsuming both games with no information (Example 9.1) and games with perfect information (e.g., the Stonier–Skowronski model). This is a direct consequence of our having specified the admissible dynamics of the players of the players first and then introduced their admissible strategies only as indices of these admissible dynamical systems. In this way we have avoid all questions on the analytical nature of strategies and their associated difficulties. Without such primarily peripheral distractions we expect our model to offer far more penetrating insight into the logical structure of a game conflict. We will to some extent justify this remark with our detailed analysis of a two-person nonzero sum quantitative game in the last few sections of this paper.

It is usual in differential games to demand uniqueness of solutions for any given strategy N-tuple $(s_1, s_2, \ldots, s_N) \in \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_N$ and any given initial condition. We, however, impose no such restriction on the general semi-dynamical systems $F_{s_1 s_2 \cdots s_N}$ in $\mathfrak{D}_{12\cdots N}$, although we do not exclude it from happening (Example 9.1). There seem to be two possible interpretations for nonunique outcomes for a given strategy N-tuple, and both may hold simultaneously. Firstly, the strategies may be too crude to guarantee uniqueness, as can be seen in the setvalued feedback strategies used by both Krasovskii [10] and Stonier [23]. Another example is the bifurcation on discontinuity manifolds in Berkovitz [1]. A second explanation for non-uniqueness is that it is an inherent feature of the system, resulting, for example, from the presence of bounded uncontrolled disturbances, e.g., [2]. (These unbounded disturbances could be attributed to a player called *Nature* whose set $\mathfrak{D}_i$ contains only one admissible dynamical system.)

The switching Axiom IV is probably the great cause of difficulty in approaching our model from the differential level. For example, in differential games with perfect information, switching introduces discontinuities into the admissible feedback strategies and thus may violate existence criteria for differential equations. It is, in fact, for this reason that switching is not allowed in Krasovskii's contingent equation games. This is, however, more a deficiency in the nature of the differential equations used to describe the game dynamics than a consequence of the game conflict itself. It does not arise in our model because we assume the existence of the game dynamics in advance. It should also be remembered that many differential models are themselves just convenient approximations of reality, so there seems little point in restricting one's attention to them alone or in considering their limitations as being absolute.

12. The Objectives of the Players

The ith player has as his objective in the game some property P_i which he wishes to secure when play commences somewhere within a certain set $\Omega_i \subset X \times R^+$ of initial events of interest to him ($i = 1, 2, \ldots, N$). He is, of course, assumed to know what his own objective is, but he need not be fully aware of those of the other players, e.g., see [21]. Conflict may then occur, depending on the interrelation of the N ordered pairs (Ω_i, P_i). It may vary from none at all (i.e., coorperative games) to total antagonism (zero sum games). A classification of some of the possibilities can be found in [22].

Whatever the objectives are P_i ($i = 1, 2, \ldots, N$), we will always assume here that $\Omega = \bigcap_{i=1}^{N} \Omega_i \neq \varnothing$ and will consider only those $(x_0, t_0) \in \Omega$, so every player will be actively involved in the game.

It is usual to classify the objectives P_i as either *qualitative* or *quantitative* [3]. In the next two sections we will consider each separately, though in general it is not necessary for all of the players in a given game to have the same kind of objective. Furthermore, since our model is primarily concerned with the dynamics of games, much of what has already been said in the literature about the objectives of players will be applicable here. Consequently we will concentrate on those aspects peculiar to our particular formulation of game dynamics.

13. Qualitative Objectives

By qualitative objectives we mean such things as reaching a given target set, ultimate boundedness within a certain region and stability of some set, e.g., [2, 19, 22, 23].

Our formulation of game dynamics is well suited to this kind of objective and allows each player to first investigate the game as a control rather than conflict problem. Without doubt any player would much prefer to achieve his desired objective P_i without having to take into account the decisions made by the other players, particularly when he is not fully aware of their objectives. He can do this by searching his set $\mathfrak{D}_i$ for an admissible dynamical system F_{s_i} which fulfils his objective P_i for play starting within the set Ω_i of initial events. The extensively developed Lyapunov theory of general semi-dynamical systems is available to aid in this search, though in practice, finding suitable Lyapunov functions will be a matter of experience and luck.

If successful in this search, Player i can then implement such a system, more or less without regard to what the other players will do. Skowronski [22] calls this situation *i-playability* and Roxin [19] calls it *i-controllability*. A detailed example of this and the use of Lyapunov theory can be found in Stonier [23] in which is considered a two-person game with one player trying to strongly equi-ultimately bound his admissible dynamical system in a certain compact subset of the state space and the other player trying to prevent this.

In general, however, we do not expect a player to be successful in such a search as it is a rather stringent situation and, in effect, means there is no conflict for him. Of far greater game-theoretic interest is the situation in which no player is successful and there is actual conflict between their desired objectives. Then we expect, in analogy with the compromise solutions of quantitative games (e.g., saddles, Nash equilibria), that with "rational" play each player should be able to secure no worse than some weaker property $P_i{}^*$. This will of course require some sort of preference ordering on possible properties P_i for each player i. At the time of writing, about all we can say is that the procedures used to find the compromise properties $P_i{}^*$ will depend very much on the particular preference orderings and game dynamics ($i = 1, 2, \ldots, N$). We can, however, illustrate it with the two-person game with no information introduced in Example 9.1.

EXAMPLE 13.1. Let $X = R$ and $\Omega_1 = \Omega_2 = X \times R^+$. Furthermore, let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be the sets of admissible general semi-dynamical systems found in Example 9.1. We refer to [8] and [15] for definitions of the stabilities used below.

For the objective P_1 of Player 1 we take the *positive asymptotic strong stability* of the set $A_1 = \{0\} \subset X$ and as the objective P_2 of Player 2 the *positive asymptotic strong stability* of the set $A_2 = \{x \in X; \mid x \mid \geqslant 10\}$.

It can be easily seen by inspection that there is no $F_u \in \mathfrak{D}_1$ for which P_1 is satisfied, but Player 1 need achieve no worse than the *positive strong stability* ($P_1{}^*$) of A_1 by playing $F_{u^*} \in \mathfrak{D}$ where $u^*(t) \equiv 1$ a.e. Similarly, there is no $F_v \in \mathfrak{D}_2$ for which P_2 is satisfied, but Player 2 need do no worse than the *positive strong stability* ($P_2{}^*$) of A_2 by playing $F_{v^*} \in \mathfrak{D}_2$ where $v(t) \equiv 1$ a.e. Neither player here can guarantee in advance doing better than his $P_i{}^*$ as

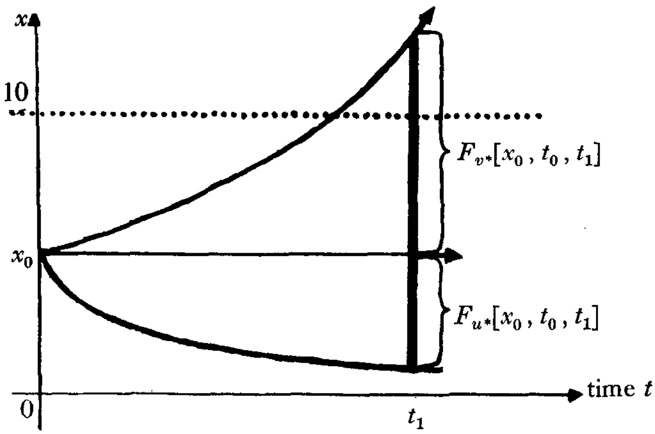

FIGURE 3

that will depend on what the other player does, but neither player need do any worse. This can be seen from Fig. 3 where we show graphically the attainability sets $F_{u^*}(x_0, t_0, t_1)$, $F_{v^*}(x_0, t_0, t_1)$ and $F_{u^*v^*}(x_0, t_0, t_1) \equiv \{x_0\}$ for any $x_0 > 0$ and $t_1 > t_0 = 0$.

14. Quantitative Objectives

Games with quantitative objectives involve the players in maximizing or minimizing payoff functionals either at a given instant of time (games of fixed duration) or at the first instant of time that the corresponding $F_{s_1 s_2 \cdots s_N}(x_0, t_0, t)$ is completely absorbed by a target set $\Theta \subset X$ (games with terminating target sets).

Since we have really only presented a new formulation of game dynamics here, the usual quantitative game-theoretic concepts such as value, saddles, and Nash equilibria will be applicable here in the appropriate circumstances (though Isaacs equation for value will not be in its present form as it is structurally dependent on a differential formulation of game dynamics). The possible multivaluedness of the general semi-dynamical systems $F_{s_1 s_2 \cdots s_N}$ in $\mathfrak{D}_{12 \cdots N}$, however, leads to hitherto unconsidered aspects of dynamical games. In Sections 15–18 we will look at a two-person quantitative game of fixed duration which superficially appears to be zero sum, but is in fact nonzero sum if any $F_{s_1 s_2 \cdots s_N}$ in $\mathfrak{D}_{12 \cdots N}$ is multivalued. More serious, though, is the situation in which some $F_{s_1 s_2 \cdots s_N}(x_0, t_0, t)$ is multivalued and only ever partially absorbed by Θ. This gives rise to some conceptual and practical complications, the resolution of which will depend on the interpretation given to the multivaluedness of $F_{s_1 s_2 \cdots s_N}$. See, for example, [2].

For any $(x_0, t_0) \in \Omega$ and $F_{s_1 s_2 \cdots s_N}$ in $\mathfrak{D}_{12\cdots N}$, a game with a terminating target set Θ is in effect over at the first instant of time

$$t_1 = t_1(x_0, t_0, s_1, s_2, \ldots, s_N) \geqslant t_0$$

for which $F_{s_1 s_2 \cdots s_N}(x_0, t_0, t_1) \subset \Theta$. As is usually done in differential games we could simply disregard what happens after time t_1 even though $F_{s_1 s_2 \cdots s_N}$ may not remain in Θ. Alternatively, we could follow Roxin [19] and modify all of the $F_{s_i} \in \mathfrak{D}_i$ $(i = 1, 2, \ldots, N)$ so that the set Θ is positively strongly invariant with respect to each of them. Roxin did this in order to be able to use his Lyapunov theorems on *finite strong stability* to see whether or not the set Θ is ever reached. In the following example we show that this procedure may cause the properties of closedness and upper semicontinuity in initial conditions of the GSDS to be violated. This greatly reduces its appeal, although we can nevertheless still use the certain duality of stability and controllability if instead we use Storey's concept of *final stability* [24] which does not require the positive strong invariance of the set Θ.

EXAMPLE 14.1. Let $X = R^2$, $\Theta = \{(x, y) \in R^2; x = 10 \text{ and } 0 \leqslant y \leqslant 10\}$ and consider the two-person game with no information governed by

$$(\dot{x}, \dot{y}) = (u, v) \tag{14.1}$$

where $0 \leqslant u \leqslant 1$ and $0 \leqslant v \leqslant 2$. The admissible strategies $\mathfrak{U} = u(\cdot)$ and $\mathfrak{B} = v(\cdot)$ are similar to those in Example 9.1.

Then if Player $1(u)$ uses $u(t) \equiv 1$ a.e., his resultant admissible GSDS F_u is

$$F_u((x_0, y_0), t_0, t) = \{t - t_0 + x_0\} \times [-2(t - t_0) + y_0, 2(t - t_0) + y_0]$$

for all $(x_0, y_0) \in R$ and $t \geqslant t_0 \geqslant 0$. We can modify this to make Θ positively strongly invariant by altering (14.1) to hold only on $R^2 \backslash \Theta$ and setting $\dot{x} = \dot{y} = 0$ on Θ. Then for, say, $x_0 = y_0 = 8$ and $t_0 = 0$, the modified attainability sets are

$$\begin{aligned} &\{t + 8\} \times [-2t + 8, 2t + 8] && \text{for} \quad 0 \leqslant t \leqslant 2, \\ &(\{10\} \times [4, 10]) \cup (\{t + 8\} \times (-2t + 4, 2t + 8]) && \text{for} \quad 2 \leqslant t. \end{aligned}$$

These sets are not closed for $t > 2$. Moreover, by examining the modified attainability sets for initial conditions (x_0, t_0) on either side of Θ, we can easily see that upper semicontinuity in initial conditions is also violated.

15. A Two-Person Quantitative Game

We will now consider in some detail a two-person quantitative game of fixed duration for which the (unspecified) sets $\mathfrak{D}_i$ ($i = 1, 2$) of admissible dynamical systems are assumed to satisfy Axioms I–III of our game model. Axiom IV is omitted as it will not be required in the sequel.

Let $T > 0$ be a fixed instant of time and let A be a nonempty closed subset of X. Then the objective P_1 of Player 1 is to minimize the distance at time T from the set A, and the objective P_2 of Player 2 is to maximize this distance, in both cases for any initial condition $(x_0, t_0) \in \Omega_1 = \Omega_2 = X \times [0, T)$.

As described so far this game seems to be zero sum, and has in fact been investigated as such on the differential level. In particular, we draw attention to the work of Krasovskii and associates [10–12, 26]) who have defined saddle points for this game and given a sufficiency condition for their existence. This condition is, however, very much dependent on the analytical form of the game differential equation and does not seem easy to verify in practice. Furthermore, we believe it contributes little to an understanding of what is actually happening in the game.

In the next two sections we will consider these saddle points of Krasovskii and derive a condition in terms of our formulation of game dynamics which is necessary and sufficient for their existence. We believe that the conceptual simplicity and ease with which this condition can be tested gives much insight into the acutal game conflict, and thus illustrates the potential of our model.

Like Krasovskii we allow the general semi-dynamical systems $F_{s_1 s_2}$ in $\mathfrak{D}_{12}$ to be multivalued, but unlike him, we show that this means the game is really nonzero sum. Consequently we find his use of the term *saddle point* potentially misleading, and in the last section we briefly consider Nash equilibria and von Stackelberg solution concepts for this game.

16. Krasovksii's Saddle Points

Krasovskii [11] has defined *saddle points* for this game as follows.

Definition 16.1. For any fixed $(x_0, t_0) \in \Omega$ a *saddle point* is a pair of admissible strategies $(s_1^*, s_2^*) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ such that

$$\rho(x, A) \leqslant \rho(y, A) \leqslant \rho(z, A)$$

for all $x \in F_{s_1^* s_2}(x_0, t_0, T)$, $y \in F_{s_1^* s_2^*}(x_0, t_0, T)$, $z \in F_{s_1 s_2^*}(x_0, t_0, T)$ and all $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$.

The geometric meaning of this is more transparent when expressed in the following equivalent form.

PROPOSITION. *Let* $(x_0, t_0) \in \Omega$ *be fixed. Then* $(s_1^*, s_2^*) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ *is a saddle point if and only if*

$$\rho^*(F_{s_1^* s_2}(x_0, t_0, T), A) \leqslant \delta(F_{s_1^* s_2^*}(x_0, t_0, T), A), \tag{16.1}$$

$$\rho^*(F_{s_1^* s_2^*}(x_0, t_0, T), A) \leqslant \delta(F_{s_1 s_2^*}(x_0, t_0, T), A), \tag{16.2}$$

both hold simultaneously for all $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$.

Proof. By (2.3) and (2.4) the following inequalities

$$\begin{aligned} \rho(x, A) &\leqslant \rho^*(F_{s_1^* s_2}(x_0, t_0, T), A), \\ \delta(F_{s_1^* s_2^*}(x_0, t_0, T), A) \leqslant \rho(y, A) &\leqslant \rho^*(F_{s_1^* s_2^*}(x_0, t_0, T), A), \\ \delta(F_{s_1 s_2^*}(x_0, t_0, T), A) &\leqslant \rho(z, A), \end{aligned} \tag{16.3}$$

are true for all $x \in F_{s_1^* s_2}(x_0, t_0, T)$, $y \in F_{s_1^* s_2^*}(x_0, t_0, T)$, $z \in F_{s_1 s_2^*}(x_0, t_0, T)$, and all $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$.

Hence, (16.1) implies that $\rho(x, A) \leqslant \rho(y, A)$ and (16.2) that $\rho(y, A) \leqslant \rho(z, A)$ for all such x, y, and z. Consequently, such an (s_1^*, s_2^*) is a saddle point.

The necessity part of the proof follows immediately from the fact that the attainability sets are all compact, and consequently, the infima and suprema over them of the distance functions in (16.1) and (16.2) are actually attained. ∎

The next proposition shows that if (s_1^*, s_2^*) is a saddle point for some initial condition $(x_0, t_0 \in \Omega$ then all of the points in the set $F_{s_1^* s_2^*}(x_0, t_0, T)$ are equidistant from A. Krasovskii calls this distance $V(x_0, t_0)$ the *value* of the game at (x_0, t_0).

PROPOSITION 16.2. *Let* $(s_1^*, s_2^*) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ *be a saddle point of the game for some fixed* $(x_0, t_0) \in \Omega$. *Then there exists a constant* $V(x_0, t_0)$ *such that*

$$V(x_0, t_0) = \delta(F_{s_1^* s_2^*}(x_0, t_0, T), A) = \rho^*(F_{s_1^* s_2^*}(x_0, t_0, T), A).$$

Proof. As (16.1) holds for all $s_2 \in \mathfrak{S}_2$, it holds in particular for $s_2^* \in \mathfrak{S}_2$, and so,

$$\rho^*(F_{s^* s^*}(x_0, t_0, T), A) \leqslant \delta(F_{s^* s^*}(x_0, t_0, T), A).$$

This combined with (16.3) gives the desired result. ∎

17. Extremal Strategies and Saddle Points

We now define *extremal strategies* for each player in this game. They are not unlike Krasovskii's extremal aiming strategies, though are defined quite differently. After establishing some of their basic properties we will use them to derive a test for the existence of saddle points of the game.

DEFINITION 17.1. (See 26). Let $(x_0, t_0) \in \Omega$ be any fixed initial condition. Then an admissible strategy

(i) $s_1^* \in \mathfrak{S}_1$ is called *extremal* for Player 1 if

$$\Gamma^0(x_0, t_0) = \rho^*(F_{s_1^*}(x_0, t_0, T), A) = \inf\{\rho^*(F_{s_1}(x_0, t_0, T), A); s_1 \in \mathfrak{S}_1\};$$

(ii) $s_2^* \in \mathfrak{S}_2$ is called *extremal* for Player 2 if

$$\Gamma_0(x_0, t_0) = \delta(F_{s_2^*}(x_0, t_0, T), A) = \sup\{\delta(F_{s_2}(x_0, t_0, T), A); s_2 \in \mathfrak{S}_2\}.$$

THEOREM 17.1. *Extremal strategies exist for both Players* 1 *and* 2 *for any initial condition* $(x_0, t_0) \in \Omega$.

Proof. By Axiom I the set $\mathfrak{D}_i$ of admissible general semi-dynamical systems is a compact subset of $\mathfrak{F}(X)$ and consequently the set $\{F_{s_i}(x_0, t_0, T); s_i \in \mathfrak{S}_i\}$ is a compact subset of the space of nonempty closed subsets of X topologized with the Hausdorff metric ($i = 1, 2$). From this compactness and the properties of the two distance functions ρ^* and δ, it follows that the infimum and supremum in Definition 17.1 are actually attained. ▮

By Corollary 10.1 and the compactness of the sets $\mathfrak{D}_i$ ($i = 1, 2$) and of the attainability sets of GSDS, it is easily seen that

$$\rho^*(F_{s_1}(x_0, t_0, T), A) = \max\{\rho^*(F_{s_1 s_2}(x_0, t_0, T), A); s_2 \in \mathfrak{S}_2\}, \tag{17.1}$$

and

$$\delta(F_{s_2}(x_0, t_0, T), A) = \min\{\delta(F_{s_1 s_2}(x_0, t_0, T), A); s_1 \in \mathfrak{S}_1\} \tag{17.2}$$

for any $(x_0, t_0) \in \Omega$ and any $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$. From this we can derive the following properties of extremal strategies.

THEOREM 17.2. *For any fixed initial condition* $(x_0, t_0) \in \Omega$ *a strategy*

(i) $s_1^* \in \mathfrak{S}_1$ *is extremal for Player* 1 *if and only if it is minimax*;

(ii) $s_2^* \in \mathfrak{S}_2$ *is extremal for Player* 2 *if and only if it is maximin.*

Proof. For (17.1) and (17.2) we have, for all $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$,

$$\begin{aligned}\Gamma^0(x_0, t_0) &= \max\{\rho^*(F_{s_1^* s_2}(x_0, t_0, T), A); s_2 \in \mathfrak{S}_2\}\\ &= \max\{\rho^*(F_{s_1 s_2}(x_0, t_0, T), A); s_2 \in \mathfrak{S}_2\},\end{aligned}$$

and

$$\begin{aligned}\Gamma_0(x_0, t_0) &= \min\{\delta(F_{s_1 s_2^*}(x_0, t_0, T), A); s_1 \in \mathfrak{S}_1\}\\ &= \min\{\delta(F_{s_1 s_2}(x_0, t_0, T), A); s_1 \in \mathfrak{S}_1\}.\end{aligned}$$

These are respectively the definitions of the minimax strategy for Player 1 and the maximin strategy for Player 2. ▮

THEOREM 17.3. *$\Gamma_0(x_0, t_0) \leqslant \Gamma^0(x_0, t_0)$ for all $(x_0, t_0) \in \Omega$.*

Proof. By Theorem 17.1, both $\Gamma_0(x_0, t_0)$ and $\Gamma^0(x_0, t_0)$ exist for any (x_0, t_0) in Ω. Now let $(x_0, t_0) \in \Omega$ be fixed and let $(s_1^*, s_2^*) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ be the corresponding extremal strategies. Then

$$\begin{aligned}\Gamma_0(x_0, t_0) &= \delta(F_{s_2^*}(x_0, t_0, T), A)\\ &\leqslant \delta(F_{s_1^* s_2^*}(x_0, t_0, T), A)\\ &\leqslant \rho^*(F_{s_1^* s_2^*}(x_0, t_0, T), A)\\ &\leqslant \rho^*(F_{s_1^*}(x_0, t_0, T), A) = \Gamma^0(x_0, t_0)\end{aligned}$$

by the definitions of the distance functions and attainability sets used. ▮

In terms of our formulation of game dynamics, it is very easy to determine the extremal strategies of the numerical values of $\Gamma_0(x_0, t_0)$ and $\Gamma^0(x_0, t_0)$ for any initial condition $(x_0, t_0) \in \Omega$. For this no assumptions are needed on the kind of information available to the players, other than knowledge of the initial condition. In fact, each player can determine his own extremal strategy and corresponding quantity Γ_0 or Γ^0 without even knowing what the objective of the other player is, in which circumstances an extremal strategy corresponds to his making the best out of the potentially worst situation.

Our next theorem gives a very simple test for the existence or not of saddle points for this game.

THEOREM 17.4. *Let $(x_0, t_0) \in \Omega$ be fixed. Then a necessary and sufficient condition for a pair of strategies $(s_1^*, s_2^*) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ to be a saddle point is that*

(i) *(s_1^*, s_2^*) are extremal strategies;*
(ii) $\Gamma_0(x_0, t_0) = \Gamma^0(x_0, t_0)$.

In this case, we write $V(x_0, t_0) = \Gamma_0(x_0, t_0) = \Gamma^0(x_0, t_0)$.

Proof. Let $(s_1{}^*, s_2{}^*) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ be a saddle point for $(x_0, t_0) \in \Omega$. Then by Propositions 16.1 and 16.2, we have for all $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$

$$V(x_0, t_0) = \delta(F_{s_1{}^*s_2{}^*}(x_0, t_0, T), A) \geqslant \rho^*(F_{s_1{}^*s_2}(x_0, t_0, T), A),$$

and

$$V(x_0, t_0) = \rho^*(F_{s_1{}^*s_2{}^*}(x_0, t_0, T), A) \leqslant \delta(F_{s_1s_2{}^*}(x_0, t_0, T), A).$$

From this and 17.1, we then have for all $s_1 \in \mathfrak{S}_1$,

$$\begin{aligned}\rho^*(F_{s_1{}^*}(x_0, t_0, T), A) &= \rho^*(F_{s_1{}^*s_2{}^*}(x_0, t_0, T), A)\\ &\leqslant \delta(F_{s_1s_2{}^*}(x_0, t_0, T), A)\\ &\leqslant \rho^*(F_{s_1s_2{}^*}(x_0, t_0, T), A)\\ &\leqslant \rho^*(F_{s_1}(x_0, t_0, T), A),\end{aligned}$$

so $s_1{}^*$ is an extremal strategy for Player 1 and $V(x_0, t_0) = \Gamma^0(x_0, t_0)$.

Similarly, using 17.2 now, we have for all $s_2 \in \mathfrak{S}_2$

$$\begin{aligned}\delta(F_{s_2{}^*}(x_0, t_0, T), A) &= \delta(F_{s_1{}^*s_2{}^*}(x_0, t_0, T), A)\\ &\geqslant \rho^*(F_{s_1{}^*s_2}(x_0, t_0, T), A)\\ &\geqslant \delta(F_{s_1{}^*s_2}(x_0, t_0, T), A)\\ &\geqslant \delta(F_{s_2}(x_0, t_0, T), A),\end{aligned}$$

so $s_2{}^*$ is an extremal strategy for Player 2 and $V(x_0, t_0) = \Gamma_0(x_0, t_0)$.

To prove sufficiency, let $(s_1{}^*, s_1{}^*) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ be extremal strategies and let $\Gamma^0(x_0, t_0) = \Gamma^0(x_0, t_0)$. Then by Definition 17.1 and the definitions of the distances ρ^* and δ, we have for all $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$,

$$\Gamma_0(x_0, t_0) = \delta(F_{s_2{}^*}(x_0, t_0, T), A) \leqslant \delta(F_{s_1s_2{}^*}(x_0, t_0, T), A),$$

and

$$\Gamma^0(x_0, t_0) = \rho^*(F_{s_1{}^*}(x_0, t_0, T), A) \geqslant \rho^*(F_{s_1{}^*s_2}(x_0, t_0, T), A).$$

Thus, by the hypothesis that $\Gamma_0(x_0, t_0) = \Gamma^0(x_0, t_0)$, we have for all $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$

$$\rho^*(F_{s_1{}^*s_2{}^*}(x_0, t_0, T), A) \leqslant \delta(F_{s_1s_2{}^*}(x_0, t_0, T), A),$$

and

$$\rho^*(F_{s_1^* s_2}(x_0, t_0, T), A) \leqslant \delta(F_{s_1^* s_2^*}(x_0, t_0, T), A),$$

and so, by Proposition 16.1, (s_1^*, s_2^*) is a saddle point. ▮

Krasovskii's sufficiency condition for this game is an analytical condition based on the game differential equation which in effect ensures that $\Gamma^0(x_0, t_0) = \Gamma^0(x_0, t_0)$. He calls such games *regular*. His work is, however, restricted to games with perfect information, and his test for regularity is by no means easy to carry out.

We conclude this section with a simple example of a game with such a saddle point.

EXAMPLE 17. Let $X = R$, $A = \{0\} \subset X$, and let the sets $\mathfrak{D}_i$ $(i = 1, 2)$ of admissible dynamics be as in Example 9.1. Then for any $(x_0, t_0) \in \Omega$, the extremal strategy for Player 1 is $u^*(t) \equiv 1$ a.e. with $\Gamma^0(x_0, t_0) = |x_0|$, and for Player 2, $v^*(t) \equiv 1$ with $\Gamma_0(x_0, t_0) = |x_0|$, too. Thus, (u^*, v^*) is a saddle point in the sense of Krasovskii. (In fact it is a saddle point in the true sense as each F_{uv} in $\mathfrak{D}_{12}$ is single-valued in this example.)

18. NASH EQUILIBRIA

Our statement in Section 15 of the objectives P_i $(i = 1, 2)$ of the players in this game is somewhat ambiguous if the attainability set $F_{s_1 s_2}(x_0, t_0, T)$ is multivalued for some $F_{s_1 s_2}$ in $\mathfrak{D}_{12}$: From which points in this set do the players measure the distances they wish to minimize or maximize?

In what follows we will hold the initial condition $(x_0, t_0) \in \Omega$ fixed. Then from the conservative nature of the saddle point concept, it seems that Player 1 should minimize the distance

$$J_1(s_1, s_2) = \rho^*(F_{s_1 s_2}(x_0, t_0, T), A)$$

of the furtherest point in $F_{s_1 s_2}(x_0, t_0, T)$ from the set A, and Player 2 should try to maximize the distance

$$J_2(s_1, s_2) = \delta(F_{s_1 s_2}(x_0, t_0, T), A)$$

of the closest point in this set A. We will take these to be the objectives P_1 and P_2 of Players 1 and 2, respectively.

Now it is always true for all strategies $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ that

$$J_2(s_1, s_2) \leqslant J_1(s_1, s_2),$$

from 2.3 and 2.4. However, equality need not hold here unless we impose further restrictions on the attainability sets $F_{s_1 s_2}(x_0, t_0, T)$. Consequently, this game will in general be nonzero sum and strictly speaking a solution concept such as *Nash equilibria* should be used instead of saddle points.

DEFINITION 18.1. (E.g., see [21].) An admissible strategy pair

$$(s_1^*, s_2^*) \in \mathfrak{S}_1 \times \mathfrak{S}_2$$

is called a *Nash equilibrium point* if for all $(s_1, s_2) \in \mathfrak{S}_1 \times \mathfrak{S}_2$ both of the following hold:

(i) $J_1(s_1^*, s_2^*) \leqslant J_1(s_1, s_2^*)$;
(ii) $J_2(s_1^*, s_2) \leqslant J_2(s_1^*, s_2^*)$.

Consequently, Krasovskii's saddle points are thus really just Nash equilibria for which

$$\Gamma_0(x_0, t_0) = J_2(s_1^*, s_2^*) = J_1(s_1^*, s_2^*) = \Gamma^0(x_0, t_0).$$

They, however, need not be saddle points in the true sense unless we impose further assumptions on the attainability sets $F_{s_1 s_2}(x_0, t_0, T)$, as in Example 17.1. For this reason, we feel that Krasovskii's terminology is potentially misleading.

Our Theorem 17.4 is thus a sufficient condition for the existence of a Nash equilibrium point for this game. We now ask if Nash equilibria need always exist when $\Gamma_0(x_0, t_0) < \Gamma^0(x_0, t_0)$? To this we conjecture a negative answer with our reasoning based on the following example.

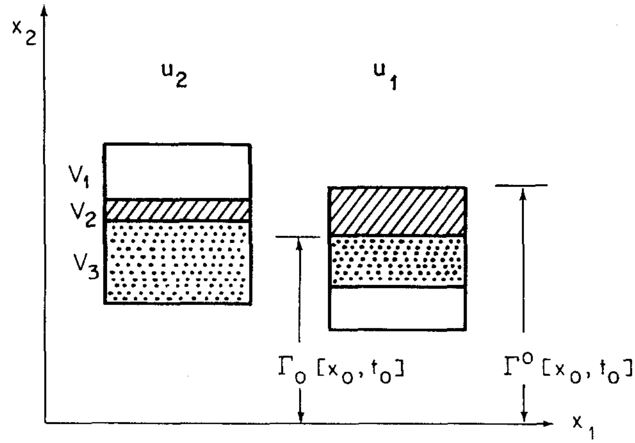

FIGURE 4

EXAMPLE 18.1. Let $X = R^2$ and $A = R \times \{0\}$. Then consider sets of admissible dynamical systems $\mathfrak{D}_1 = \{F_{u_1}, F_{u_2}\}$ and $\mathfrak{D}_2 = \{F_{v_1}, F_{v_2}, F_{v_3}\}$ with attainability sets at time T for some initial condition $(x_0, t_0) \in \Omega$ as illustrated in Fig. 4.

The extremal strategies are u_1 and v_2, and $\Gamma_0(x_0, t_0) < \Gamma^0(x_0, t_0)$. It is, however, easy to see that no pair of strategies (u_i, v_j) is a Nash equilibrium point, in particular the extremal pair (u_1, v_2).

For this situation the *von Stackelberg* solution concept comes to mind. See [21] for details. It is easily seen that with Player 1 as the leader, the extremal pair (u_1, v_2) is the von Stackelberg strategy pair. With Player 2 as the leader, the von Stackelberg strategy pair is (u_2, v_2) and both players receive a better payoff in this case. Player 1 is, however, not playing his extremal strategy here, so there needs to be some enforcement rule in the game to prevent Player 2 from taking advantage of this.

APPENDIX

The fundamental objects in our model of game dynamics are the sets $\mathfrak{D}_i \subset \mathfrak{F}(X)$ of admissible general semi-dynamical systems. The indices $s_i \in \mathfrak{S}_i$ of the $F_{s_i} \in \mathfrak{D}_i$ are then called the strategies of the ith player.

Now every motion of such an $F_{s_i} \in \mathfrak{D}_i$ can also be considered indexed, albeit implicitly, with s_i. Then what our operation of motion-intersection does in reclaiming the actual game dynamics

$$F_{s_1 s_2 \cdots s_N} = F_{s_1} \wedge F_{s_2} \wedge \cdots \wedge F_{s_N}$$

for a given strategy N-tuple $(s_1, s_2, \ldots, s_N) \in \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_N$, is in effect to separate out all those motions common to these N general semi-dynamical systems F_{s_i} and to index them implicitly with the N-tuple $(s_1, s_2, \ldots, s_N)$. When nothing else is known about the dynamics of a game this procedure seems most reasonable, if indeed the only means available of reclaiming the actual game dynamics.

The referee has drawn our attention to a serious conceptual difficulty which may arise if there are natural sets of strategies underlying the sets of admissible general semi-dynamical systems, as in differential games. For example consider the two-person differential game with no information

$$\dot{x} = uv \tag{A.1}$$

where u and v are constrained to $[0, 1]$. Then for $u^*(t) = v^*(t) = 1$ a.e., we have for each $x_0 \geq 0$ and $t \geq t_0 \geq 0$,

$$F_{u^*}(x_0, t_0, t) = F_{v^*}(x_0, t_0, t) = [x_0, x_0 + t - t_0].$$

These two general semi-dynamical systems have the motion-intersection

$$F_{u*} \wedge F_{v*}(x_0, t_0, t) = [x_0, x_0 + t - t_0],$$

which is considerably larger than the unique trajectory

$$F_{u*v*}(x_0, t_0, t) = \{x_0 + t - t_0\}$$

obtained by inserting the open-loop startegy pair (u^*, v^*) into (A.1). All the other motions of $F_{u*} \wedge F_{v*}$ are generated by pairs of open-loop strategies (u^*, v) and (u, v^*) where $u \neq u^*$ and $v \neq v^*$. The operation of motion-intersection is unable to distinguish these motions from those generated by (u^*, v^*).

For such games we should thus keep track of which strategy N-tuple $(s_1, s_2, ..., s_N)$ a given motion is due; in effect, we should retain the actual $F_{s_1 s_2 \cdots s_N}$.

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